CYCLES OF EVERY LENGTH AND ORIENTATION IN RANDOMLY PERTURBED DIGRAPHS

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Abstract. In 2003, Bohman, Frieze, and Martin initiated the study of randomly perturbed graphs and digraphs. For digraphs, they showed that for every $\alpha > 0$, there exists a constant C such that for every n-vertex digraph of minimum semi-degree at least αn, if one adds Cn random edges then asymptotically almost surely the resulting digraph contains a consistently oriented Hamilton cycle. We generalize their result, showing that the hypothesis of this theorem actually asymptotically almost surely ensures the existence of every orientation of a cycle of every possible length, simultaneously. Moreover, we prove that we can relax the minimum semi-degree condition to a minimum total degree condition when considering orientations of a cycle that do not contain a large number of vertices of indegree 1.

1. Introduction

Hamilton cycles are one of the most studied objects in graph theory, and several classical results measure how 'dense' a graph needs to be to force a Hamilton cycle. In particular, in 1952 Dirac [8] proved that every n-vertex graph with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle; the minimum degree condition here is best possible.

The Hamiltonicity of directed graphs has also been extensively investigated since the 1960s. A *directed graph*, or *digraph*, is a set of vertices together with a set of ordered pairs of distinct vertices. We think of a digraph as a loop-free multigraph, where every edge is given an orientation from one endpoint to another, and there is at most one edge oriented in each of the two directions between a pair of vertices. An *oriented graph* is a digraph with at most one directed edge between every pair of vertices. An edge from vertex u to vertex v is represented as $\overrightarrow{uv}$ or $\overleftarrow{vu}$. In the digraph setting, there is more than one natural analog of the minimum degree of a graph. The *minimum semi-degree* $\delta^0(D)$ of a digraph D is the minimum of all the in- and outdegrees of the vertices in D; the *minimum total degree* $\delta(D)$ is the minimum number of edges incident to a vertex in D. Ghouila-Houri [12] proved that every strongly connected n-vertex digraph D with minimum total degree $\delta(D) \geq n$ contains a *consistently oriented* Hamilton cycle, that is, a cycle $(v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$

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with edges $\overrightarrow{v_iv_{i+1}}$ for all $i \in [n]$. Note that there are n-vertex digraphs D with $\delta(D) = 3n/2 - 2$ that do not contain a consistently oriented Hamilton cycle, so the strongly connected condition in Ghouila-Houri's theorem is necessary.

An immediate consequence of Ghouila-Houri's theorem is that having minimum semi-degree $\delta^0(D) \geq n/2$ forces a consistently oriented Hamilton cycle, and this is best possible. After earlier partial results [13, 14], DeBiasio, Kühn, Molla, Osthus, and Taylor [6] proved that this minimum semi-degree condition in fact forces all possible orientations of a Hamilton cycle, except for the *anti-directed* Hamilton cycle, that is, a cycle $(v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ with edges $\overrightarrow{v_iv_{i+1}}$ for all odd $i \in [n]$ and $\overleftarrow{v_iv_{i+1}}$ for all even $i \in [n]$, where n is even. Earlier, DeBiasio and Molla [7] showed that the minimum semi-degree threshold for forcing the anti-directed Hamilton cycle is in fact $\delta^0(D) \geq n/2 + 1$.

There has also been interest in Hamilton cycles in random digraphs: the *binomial random digraph* $D(n, p)$ is the digraph with vertex set $[n]$, where each of the $n(n-1)$ possible directed edges is present with probability p, independently of all other edges. Recently, Montgomery [21] determined the sharp threshold for the appearance of any fixed orientation of a Hamilton cycle H in $D(n, p)$, thereby answering a conjecture of Ferber and Long [11] in a strong form. Depending on the orientation of H, the threshold here can vary from $p = \log n/2n$ to $p = \log n/n$.

In this extended abstract, we consider arbitrary orientations of Hamilton cycles in the randomly perturbed digraph model. Introduced in both the undirected and directed setting by Bohman, Frieze, and Martin [2], this model starts with a dense (di)graph and then adds m random edges to it. The overarching question now is how many random edges are required to ensure that the resulting (di)graph *asymptotically almost surely (a.a.s.)* satisfies a given property, that is, with probability tending to 1 as the number of vertices n tends to infinity. For example, Bohman, Frieze, and Martin [2] proved that for every $\alpha > 0$, there is a $C = C(\alpha)$ such that if we start with an arbitrary n-vertex graph G of minimum degree $\delta(G) \geq \alpha n$ and add Cn random edges to it, then a.a.s. the resulting graph is Hamiltonian. Furthermore, given a constant $0 < \alpha < 1/2$, in a complete bipartite graph with part sizes αn and $(1-\alpha)n$, a linear number of random edges are needed to ensure Hamiltonicity. Thus their result is best possible up to the dependence of C on α. Subsequently, there has been a significant effort to improve our understanding of randomly perturbed graphs. See, e.g., [15, Section 1.3] and the references within for a snapshot of some of the results in the area.

Bohman, Frieze, and Martin [2] also proved the analogous result for consistently oriented Hamilton cycles in the randomly perturbed digraph model. Their result is also best possible up to the dependence of C on α, for similar reasons as the undirected setting.

Theorem 1.1 (Bohman, Frieze, and Martin [2]). *For every $\alpha > 0$, there is a $C = C(\alpha)$ such that if D_0 is an n-vertex digraph of minimum semi-degree $\delta^0(D_0) \geq \alpha n$, then $D_0 \cup D(n, C/n)$ a.a.s. contains a consistently oriented Hamilton cycle.*

A notion closely related to Hamiltonicity is *pancyclicity*, which is when a (di)graph contains cycles of every possible length. Bondy [3] generalized Dirac's theorem, showing that if $\delta(G) \geq n/2$ then G is pancyclic or $K_{n/2,n/2}$. Shortly after, Bondy [4] proposed his famous meta-conjecture that any 'non-trivial' sufficient condition for Hamiltonicity should be a sufficient condition for pancyclicity, up to a small number of exceptional graphs. Krivelevich, Kwan, and Sudakov [17] generalized Theorem 1.1 in this way, showing that the same conditions as in Theorem 1.1 imply that the randomly perturbed digraph contains consistently oriented cycles of every length.

Theorem 1.2 (Krivelevich, Kwan, and Sudakov [17]). *For every $\alpha > 0$, there is a $C = C(\alpha)$ such that if D_0 is an n-vertex digraph of minimum semi-degree $\delta^0(D_0) \geq \alpha n$, then $D_0 \cup D(n, C/n)$ a.a.s. contains a consistently oriented cycle of every length between 2 and n.*

The original rotation-extension-type proofs of Theorems 1.1 and 1.2 only guarantee consistently oriented cycles. Our main result is a generalization of Theorem 1.2 to allow for all orientations of a cycle of every possible length. Moreover, we find all these cycles simultaneously, i.e., $D_0 \cup D(n, C/n)$ a.a.s. contains all of them. This last property is an example of *universality*, a notion both well-studied in the random graph (e.g., [9, 21]) and randomly perturbed (e.g., [5, 22]) settings.

Theorem 1.3. *For every $\alpha > 0$, there is a $C = C(\alpha)$ such that if D_0 is an n-vertex digraph of minimum semi-degree $\delta^0(D_0) \geq \alpha n$, then $D_0 \cup D(n, C/n)$ a.a.s. contains every orientation of a cycle of every length between 2 and n.*

Theorem 1.3 is best possible in the sense that one really needs to add a linear number of random edges to D_0. Indeed, similarly as before, let D be the complete bipartite digraph with part sizes αn and $(1 - \alpha)n$ (where $0 < \alpha < 1/2$). Then one needs to add a linear number of edges to D to ensure a Hamilton cycle of *any* orientation.

It is also natural to try and generalize Theorem 1.1 in another direction, by relaxing the minimum semi-degree condition to a total degree. Unfortunately, this cannot be true for a Hamilton cycle H in which all but $o(n)$ vertices have in- and outdegree 1. Indeed, given $0 < \alpha < 1/2$, let D be the n-vertex digraph which consists of vertex classes S and T of sizes αn and $(1-\alpha)n$ respectively, and whose edge set consists of all possible edges with their startpoint in S and their endpoint in T. Then whilst $\delta(D) = \alpha n$, given any constant C, with probability bounded away from 0, $D \cup D(n, C/n)$ contains a linear number of vertices with outdegree 0 and a linear number of vertices with indegree 0, so it will not contain H.

On the other hand, we show that this type of orientation of a Hamilton cycle is the only one we cannot guarantee. That is, our desired relaxation is possible for all orientations of a Hamilton cycle that contain a linear number of vertices of in- or outdegree 2.

Theorem 1.4. *For every $\alpha, \eta > 0$, there is a $C = C(\alpha, \eta)$ such that if D_0 is an n-vertex digraph of minimum total degree $\delta(D_0) \geq 2\alpha n$, then $D_0 \cup D(n, C/n)$ a.a.s. contains every orientation of a cycle of every length between 2 and n that contains at most $(1 - \eta)n$ vertices of indegree 1.*

The proof of Theorem 1.4 has the same core ideas as the proof of Theorem 1.3, but there are additional complications and technicalities that come with the weakened degree condition. We prove these two theorems in [1]. In the next section we highlight some of the ideas from the proof of Theorem 1.3.

Notation. We write $\overleftrightarrow{uv}$ if $\overrightarrow{uv}$ and $\overleftarrow{uv}$ are edges and call $\overleftrightarrow{uv}$ a *bidirected edge*. A *bidirected path* is a digraph obtained from an undirected path by replacing each edge uv with a bidirected edge $\overleftrightarrow{uv}$. An *oriented path* is a digraph obtained from an undirected path by replacing each edge uv with a single directed edge; either $\overrightarrow{uv}$ or $\overleftarrow{uv}$. Given an oriented or bidirected path $P = (u_1, \ldots, u_k)$ we call u_1 its *startpoint* and u_k its *endpoint*, distinguishing it from the path $(u_k, \ldots, u_1)$.

2. Some ideas in the proof of Theorem 1.3

Our goal is to show that for a given orientation $\mathcal{C}$ of a cycle, $D_0 \cup D(n, C/n)$ contains $\mathcal{C}$ with probability at least $1 - e^{-n}$. Theorem 1.3 then follows from a union bound over all choices of $\mathcal{C}$, of which there are trivially at most $n2^n$. For the rest of this section we consider only spanning $\mathcal{C}$, as the non-spanning cycle case follows easily from the machinery we set up to deal with arbitrary orientations of a Hamilton cycle.

Let $D^*(n,p)$ denote the random digraph with vertex set $[n]$ where each possible pair of edges $\overrightarrow{uv}$ and $\overleftarrow{uv}$ are included together, independently of other edges, with probability p. In this way $D^*(n,p)$ is the same as the binomial random graph $G(n,p)$ where we replace every undirected edge with a bidirected edge. Via a coupling argument from [18, 21], to prove that $D_0 \cup D(n, C/n)$ contains $\mathcal{C}$ with probability at least $1-e^{-n}$, it suffices to show that $D_0 \cup D^*(n, C/n)$ contains $\mathcal{C}$ with probability at least $1-e^{-n}$. This latter goal will be achievable as we only need to access the randomness in $D^*(n, C/n)$ through a simple pseudorandom property that is easily shown to hold with probability at least $1-e^{-n}$.

Our argument applies the absorbing method, a technique that was introduced systematically by Rödl, Ruciński, and Szemerédi [23], but that has roots in earlier work (see, e.g., [16]).

2.1. **Global absorbers.** For our problem, a 'global absorber' in $D_0 \cup D^*(n, C/n)$ is a structure A on a small (but linear size) vertex set with the property that for every sufficiently small set of vertices R, $A \cup R$ contains an oriented path on $|V(A) \cup R|$ vertices with prescribed startpoint and endpoint in R, and so that crucially, this oriented path is a segment of our desired orientation of a Hamilton cycle $\mathcal{C}$. If we can obtain such a structure A, then we can proceed as follows: by applying the pseudorandom property of $D^*(n, C/n)$ we find a bidirected path Q in $D^*(n, C/n)$ disjoint from A that covers almost all of the vertices not in A. Let R be the set of vertices consisting of the startpoint x and endpoint y of Q, together with all those vertices not in Q or A. Using the absorbing property of A we ensure that there is an oriented path Q_R on $V(A) \cup R$ with startpoint y and endpoint x, so that Q_R is a segment of $\mathcal{C}$. Joining the startpoints and endpoints of Q and Q_R, we obtain our desired orientation of a Hamilton cycle $\mathcal{C}$.

2.2. **Montgomery's absorbing method.** Montgomery [19, 20] introduced an approach to absorbing that has already found a number of applications, for example, to spanning trees in random graphs [19], decompositions of Steiner triple systems [10], and tilings in randomly perturbed graphs [15]. The basic idea of the method is to build a global absorber using a special graph H_m as a framework. The bipartite graph H_m has a bounded maximum degree with vertex classes $X \cup Y$ and Z, and has the property that if one deletes *any* set of vertices of a given size from X, then the resulting graph contains a perfect matching.

Roughly speaking, a global absorber is usually built from H_m as follows: every edge xy in H_m is 'replaced' with a 'local absorber' A_{xy} in such a way that all such absorbers A_{xy} are vertex-disjoint. Here a local absorber A_{xy} is some small gadget that can absorb a certain (constant size) set of vertices S_{xy} associated with x and y.

A reason why this approach has found many applications is that, in some sense, it allows one to construct a global absorber in the case when one can only find 'few' local absorbers, where what is meant by 'few' here depends on the precise setting.

In the proofs of Theorems 1.3 and 1.4 in [1] we use H_m again as a framework to build a global absorber. The reason we use H_m, however, is different from most applications of the method (although morally the reason is similar to why Montgomery used this method in [19]). In our case we will replace *every* edge in H_m incident to $z \in Z$ with the *same* local absorbing gadget A_z. Here A_z is not designed to absorb a fixed set of vertices like before; rather, it has some local flexibility about what vertices it will absorb. The idea is that constructing the global absorber in this way gives us the flexibility to know in advance precisely which (constant size) set of vertices of $\mathcal{C}$ an absorbed vertex w can play the role of. Having this 'advanced warning' about what vertices along $\mathcal{C}$ w can play the role of turns out to be a crucial property of our global absorber; see [1, Section 2] for more details.

Acknowledgments

Much of the research related to this extended abstract was carried out during a visit by the fourth and fifth authors to the University of Illinois at Urbana-Champaign. The authors are grateful to the BRIDGE strategic alliance between the University of Birmingham and the University of Illinois at Urbana-Champaign, which partially funded this visit.

Data availability statement. A full paper containing the proofs of our results can be found on arXiv [1].

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